SOLVING QUADRATIC EQUATIONS

(By the new method called The Diagonal Sum Method)

A quadratic equation in one variable has as standard form: $ax^2 + bx + c = 0$. Solving it means finding the values of x that make the equation true.

Beyond the 4 known existing solving methods (quadratic formula, factoring, completing the square, and graphing), there is a new solving method, called Diagonal Sum Method, recently presented in book titled ”New methods for solving quadratic equations and inequalities” (Amazon e-book 2010). This new solving method can directly obtain the 2 real roots without having to factor the equation.

It is a trial and error method, same as the factoring one, but it reduces the number of permutations in half by using a Rule of Signs for Real Roots. It is fast, convenient and is applicable whenever the equation is factorable. If this method fails to get the answer, then the equation can't be factored, and consequently, the quadratic formula must be used. There is a new **improved quadratic formula**, developed in the above mentioned book, that is easier to understand and remember, since students can relate the formula to the parabola graph of the given quadratic equation, considered as a function.

Innovative concept of the Diagonal Sum Method.

Direct finding 2 real roots, in the form of 2 fractions, knowing their sum ($-b/a$) and their product (c/a).

Recall the Rule of Signs for Real Roots.

It is needed to know in advance the sign status of the 2 real roots in order to reduce the number of permutations, or test cases. The Rule of Signs states as follows:

- If a and c have opposite signs, the 2 real roots have opposite signs.
  Example: The equation $5x^2 – 14x – 3 = 0$ has 2 roots that have opposite signs
- If a and c have same sign, the real roots have same sign and it can be further possible to know if both real roots are positive or negative

a. If a and b have opposite signs, both real roots are positive.
   Example: The equation $x^2 – 39x + 108 = 0$ has 2 real roots both positive

b. If a and b have same sign, both roots are negative.
   Example: The equation $x^2 + 27x + 50 = 0$ has 2 real roots both negative

More examples of using the Rule of Signs:

The equation $-6x^2 + 7x + 20 = 0$ has 2 real roots that have opposite signs
The equation $12x^2 – 113x + 253 = 0$ has 2 real roots, both positive.
The equation $21x^2 + 50x + 24 = 0$ has 2 real roots, both negative.

The diagonal sum of a pair of 2 real roots.

Given a pair of 2 real roots ($\frac{c1}{a1}$, $\frac{c2}{a2}$).

Their product is ($\frac{c}{a}$) with c1.c2 = c, and with a1.a2 = a. Their sum is equal to ($\frac{-b}{a}$).

The sum: $\frac{c1}{a1} + \frac{c2}{a2} = \frac{(c1a2 + c2a1)}{a1.a2} = \frac{-b}{a}$

The sum (c1a2 + c2a1) is called the **diagonal sum** of a root-pair.

Rule for the Diagonal Sum.

The diagonal sum of a **true** root-pair must equal to (-b). If it equals to b, then it is the negative of the solution. If the constant a is negative, the above rule is reversal in sign.

Tips for quick solving special quadratic equations.

- **Tip 1**: When a + b + c = 0, one real root is 1 and the other is (c/a)
  Example: The equation 3x^2 – 5x + 2 = 0 has 2 real roots: 1 and 2/3
- **Tip 2**: When a – b + c = 0, one real root is -1 and the other is (-c/a)
  The equation 5x^2 – 2x – 7 = 0 has 2 real roots: -1 and 7/5

SOLVING QUADRATIC EQUATIONS IN VARIOUS CASES USING THE DIAGONAL SUM METHOD.

Depending on the values of the constants a and c, solving quadratic equations may be simple or complicated.

A. When a = 1. Solving the quadratic equation type: x^2 + bx + c = 0.

In this special case, the diagonal sum becomes the sum of the 2 real roots. Solving results in finding 2 numbers knowing their sum (–b) and their product (c). Solving this type of quadratic equations by the Diagonal Sum Method doesn’t need factoring.

Example 1. Solve: x^2 - 21x – 72 = 0.

Solution. The Rule of Signs indicates the roots have opposite signs. Write the factor-pairs of c = -72. They are: (-1, 72) (-2, 36) (-3, 24)…Stop here! The sum of the 2 roots in this pair is 21 = -b. The 2 real roots are: -3 and 24.

Note. There are factor-pairs with opposite signs: (1, -72) (2, -36)…but they can be ignored since they will give the corresponding opposite diagonal sums. By convention, always put the negative sign in front of the first number of the pair.

Example 2. Solve: $-x^2 - 26x + 56 = 0$.

Solution. Roots have opposite signs. The constant a is negative. Write factor-pairs of ac = -56: (-1, 56) (-2, 28)…Stop here! This sum is 26 = -b. According to the Rule for the Diagonal Sum, when a is negative the sum must equal to (b). The true root-pair is the negative of the pair (-2, 28). The 2 real roots are: 2 and -28.

Example 3. Solve: $x^2 + 27x + 50 = 0$.

Solution. Both roots are negative. Write factors-sets of c = 50: (-1, -50) (-2, -25)…Stop here! This sum is -27 = -b. The 2 real roots are -2 and -25.

Example 4. Solve $x^2 - 39x + 108 = 0$.

Solution. Both roots are positive. Write factors-set of 108 = c.
They are: (1, 108) (2, 54) (3, 36)…Stop! This sum is 3 + 36 = 39 = -b.
The 2 real roots are: 3 and 36.

B. When a and c are prime numbers.

The Diagonal Sum Method directly selects the probable root-pairs from values of the constants a and c and it in the same time applies the Rule of Signs to these pairs.
When a and c are both **prime** numbers, the number of probable root-pairs is usually limited to one, except the cases of Tip 1 (or Tip 2).

Example 5. Solve: $7x^2 + 90x - 13 = 0$.

Solution. Roots have opposite signs. Select the probable root-pairs. The numerator contains the unique factor-pair of c (-1, 13). The denominator contains the unique factor-pair of a (1, 7). Permutation should be done to the denominator that keeps always positive.
Unique root-pair: $\left(\frac{-1}{7}, \frac{13}{1}\right)$ The diagonal sum: -1 + 91 = 90 = b
Since the diagonal equals to b, the answers are the negative of this pair: 1/7 and -13.

Note: The other root-pair (-1/1, 13/7) can be ignored since -1 is not a real root (Tip 2)

Example 6. Solve: $17x^2 + 324x + 19 = 0$.

Solution. Both roots are negative. The constants a, c are both prime numbers.

Write down the unique root-pair: $\left(\frac{-1}{17}, \frac{-19}{1}\right)$. Its diagonal sum is: -323 -1 = -324 = -b.
The 2 real roots are: -1/17 and -19.

C. When a and c are small numbers

When a and c are small numbers and contain themselves one or 2 factors, we may write down all probable root-pairs, then use mental math to calculate all diagonal sums and find the one that fits. Stop calculation when the diagonal sum matches –b (or b).

Example 7. Solve: 7x^2 – 57x + 8 = 0.

Solution. Both roots are positive. The constant c = 8 has 2 factor-pairs: (1, 8), (2, 4).

All probable root-pairs: $\left(\frac{1}{7}, \frac{8}{1}\right)$ $\left(\frac{2}{1}, \frac{4}{7}\right)$ $\left(\frac{2}{7}, \frac{4}{1}\right)$

Diagonal sums: (57 = -b) Answers: 1/7 and 8.

Example 8. Solve: 6x^2 – 19x – 11 = 0.

Solution. Roots have opposite signs. The constant a = 6 has 2 factor-pairs: (1, 6), (2, 3)

Probable root-pairs: $\left(\frac{-1}{6}, \frac{11}{1}\right)$ $\left(\frac{-1}{2}, \frac{11}{3}\right)$ $\left(\frac{-1}{3}, \frac{11}{2}\right)$

Diagonal sums: (66 – 1 = 65) (19 = -b) Answers: -1/2 and 11/3

Note: There are other opposite root-pairs $\left(\frac{1}{6}, \frac{-11}{1}\right)$, $\left(\frac{1}{2}, \frac{-11}{3}\right)$,… but they can be ignored since they are just the negative of the probable root-pairs.

Example 9. Solve: -11x^2 + 26x + 21 = 0

Solution. Roots have opposite signs, and a is negative. Probable root-pairs:

$\left(\frac{-7}{11}, \frac{3}{1}\right)$ $\left(\frac{-7}{1}, \frac{3}{11}\right)$

DS (26 = b) Answer: -7/11 and 3

D. When the constant a is negative.

Option 1. When a is negative, you can change the equation to the one with constant a positive by putting (-1) in common factor.

Example 10. Solve: – 11x^2 + 26x + 21 = 0 (1)

Solution. Make the constant a positive by adding the factor (-1):

(-1)(11x^2 – 26x - 21) = 0 (2)

Now, solve the equation (2) as usual. Remember that both Equation (1) and Equation (2) have the same root-pair.

Probable root-pairs:	$(-\frac{3}{1}, \frac{7}{11})$	$(-\frac{3}{11}, \frac{7}{1})$
DS	(-26 = b)	

Since the diagonal sum equals to b while a is positive, then the answers are: 3 and -7/11

Option 2. When the constant a is negative, you can directly solve the equation by the new method, but you must reverse the matching rule: the diagonal sum should equal to (b), instead of (-b).

<u>Example 11</u>. Solve: - 7x^2 + 34x + 5 = 0

Solution. Real roots have opposite signs. Constant a is negative. Probable root-pairs:

	$(\frac{-1}{1}, \frac{5}{7})$	$(\frac{-1}{7}, \frac{5}{1})$	
DS	(Ignore)	(34 = b)	Answers: -1/7 and 5

<u>Example 12</u>. Solve: - 5x^2 – 69x + 14 = 0

Solution. Roots have opposite signs. Probable root-pairs:

	$(\frac{-2}{1}, \frac{7}{5})$	$(\frac{-2}{5}, \frac{7}{1})$	$(-\frac{1}{5}, \frac{14}{1})$	
DS			(69 = -b)	Answers: 1/5 and -14

<u>Example 13</u>. Solve – 8x^2 + 30x – 7 = 0.

Solution. Both real roots are positive.

	$(\frac{1}{8}, \frac{7}{1})$	$(\frac{1}{2}, \frac{7}{4})$	$(\frac{1}{4}, \frac{7}{2})$	
DS			(30 = b)	Answers: 1/4 and 7/2

E. When a and c are large numbers and contain themselves many factors.

These cases are considered complicated because they involve many permutations in the solving process. In these cases, to make sure that no root-pairs may be omitted, it is advised to write all the combinations of c/a in an all-options setup. In this setup, the numerator contains all factor-pairs of c. The denominator contains all factor-pairs of a.

Example 14. Solve: 8x^2 + 13x – 6 = 0.

Solution. Roots have opposite signs. Write the all-options setup:

Factor-pairs of c = -6: (-1, 6) (-2, 3)
Factor-pairs of a = 8: (1, 8) (2, 4)

Now, you can use mental math, or a calculator, to calculate all diagonal sums and find the one that fits. It is the option (-2, 3)/(1, 8) that gives 2 probable root-pairs:

$\left(\frac{-2}{1}, \frac{3}{8}\right)$ and $\left(\frac{-2}{8}, \frac{3}{1}\right)$. The first set gives as diagonal sum: -16 + 3 = -13 = -b.

The 2 real roots are: -2 and 3/8.

Example 15. Solve: 45x^2 – 74x – 55 = 0.

Solution. Roots have opposite signs. Write the all-options setup:

Numerator. Factor-pairs of c = -55: (-1, 55) (-5, 11)
Denominator. Factor-pairs of a = 45: (1, 45) (3, 15) (5, 9)

You can use mental math, or a calculator, to calculate all diagonal sums and find the one that fits. You may also proceed by elimination. First, eliminate the options (-1, 55) / (1, 45) (3, 15) because they give large diagonal sums, compared to b = -74.

The remainder option (-5 , 11) / (5, 9), gives the unique root-pair $\left(\frac{-5}{9}, \frac{11}{5}\right)$

Its diagonal sum is -25 + 99 = 74 = -b. The 2 real roots are: -5/9 and 11/5.

Example 16. Solve: 12x^2 – 272x + 45 = 0.

Solution. Both roots are positive. Write all-options setup:

Factor-pairs of c = 45: (1, 45) (3, 15) (5, 9)
Factor-pairs of a = 12; (1, 12) (2, 6) (3, 4)

First, eliminate the options linked to (1, 12) and (3, 4) since they give odd-number-diagonal-sums while b is an even number. Secondly, look for a large-number diagonal-sum (-272).

The fitted option should be (1, 45) / (2, 6), that gives the 2 real roots: 1/6 and 45/2.

Example 17. Solve: 45x^2 – 172x + 36 = 0.

Solution. Both roots are positive. Write all-options setup:

(1, 36) (2, 18) (3, 12) (6, 6)
(1, 45) (3, 15) (5, 9)

First, eliminate options (1, 36) (3, 12) since they give odd-number-diagonal-sums. Then, eliminate options (6, 6)/(3, 15) because they give the diagonal-sums that are multiple of 3. This would make the given equation to be simplified by 3, and that is not the wanted solution. The remainder options are (2, 18)/(1, 45) and (2, 18)/(5, 9). The second option

gives 2 probable root-pairs: $(\frac{2}{5}, \frac{18}{9})$ and $(\frac{2}{9}, \frac{18}{5})$.

The second pair gives as diagonal sum 172 = b. The 2 real roots that are: 2/9 and 18/5.

Example 18. Solve: 12x^2 + 5x – 72 = 0.

Solution. Roots have opposite signs. Write all-options setup:

Numerator: (-1, 72) (-2, 36) (-3, 24) (-4, 18) (-6, 12) (-8, 9)
Denominator: (1, 12) (2, 6) (3, 4)

First, eliminate the options linked to (-2, 36) (-4, 18) (-6, 12)/(2, 6) since they give even-number diagonal sums (while b is odd). Then, eliminate options (-1, 72) (-3, 24) /(1, 12) because they give large-number diagonal sums (while b = 5). The remainder option is

(-8,9)/(3,4). This gives two probable root-pair: $(\frac{-8}{3}, \frac{9}{4})$ and $(\frac{-8}{4}, \frac{9}{3})$.

The first pair gives as diagonal sum (27 – 32 = -5 = -b). The 2 real roots are: -8/3 and 9/4.

Example 19. Solve: 24x^2 + 59x + 36 = 0.

Solution. Both roots are negative. Write the all-options setup.

Numerator: (-1, -36) (-2, -18) (-3, -12) (-4, -9) (-6, -6)
Denominator: (1, 24) (2, 12) (3, 8) (4, 6)

First, eliminate options (-2, -18) (-6, -6)/ (2, 12) (4, 6) because they give even-number-diagonal-sums (while b is odd). Then, eliminate options (-1, -36) (-2, -18) /(1, 24) since they give large diagonal-sums (while b = -59).

The remainder option is the pair $\frac{(-4, -9)}{(3, 8)}$

This pair gives two real roots: -4/3 and -9/8.

Notes: Theses are the common practices to eliminate no-fitted options:

1. Eliminate the options that give extreme diagonal sums (too large or too small) as compared with the value of constant b.

2. If b is an odd number, eliminate the linked options, that give even-number diagonal sums, to such factor-pairs like: (-2, 4), (4, 10), (-4, -6),…,

Example. Solve: $8x^2 - 35x + 12 = 0$.

Solution: Both real roots are positive. Write all-options setup: $\frac{(1, 12)(2, 6)(3, 4)}{(1, 8)(2, 4)}$

First eliminate the pairs (2, 6) and (2, 4) since they give even number diagonal sums.
Then eliminate the option (1, 12)/(1, 8) since it gives large diagonal sum (while b is 35).
The remainder option (3, 4)/(1, 8) leads to probable root-pairs: $(\frac{3}{1}, \frac{4}{8})$ and $(\frac{3}{8}, \frac{4}{1})$
The second pair gives as diagonal sum 35 = -b. The real roots are 3/8 and 4.

3. If b is an even number, eliminate the linked options, that give odd-number diagonal sums, to such factor-pairs like: (-1, 4); (3, 6); (1, -8)….

Example . Solve: $8x^2 + 2x - 15 = 0$

Solution. Roots have opposite signs. Write all-options setup: $\frac{(-1, 15)(-3, 5)}{(1, 8)(2, 4)}$
First, eliminate the linked options to the pair (1, 8) since they give odd-number diagonal sums (while b is even). Next, eliminate the linked options to (-1, 15) since they give large diagonal sums. The remainder option (-3, 5)/(2, 4) leads to probable root-pairs

$(\frac{-3}{2}, \frac{5}{4})$ $(\frac{-3}{4}, \frac{5}{2})$ The diagonal sum of first pair is (-2 = -b).
Answers: -3/4 and 5/2.

Comments.

You may ask a good question: "Why to do these complicated operations while I can quickly get the answer by using the quadratic formula with a calculator?"
Here is also a good answer: Performing these operations helps fulfill the goal of learning math that is to improve logical thinking and deductive reasoning. Imagine a situation in which you have to solve these complicated equations by the quadratic formula without a calculator, during some tests/exams for examples. It would be a boring hard work and it won't even guarantee a correct answer that may be in decimals, while the 2 true real roots are in the form of 2 fractions.

(This article was written by Nghi H. Nguyen, the co-author of the new Diagonal Sum Method)